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FIELDS OF FRACTIONS FOR GROUP ALGEBRAS OF FREE GROUPS

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ABSTRACT. Let KF be the group algebra over the commutative field K of the free group F. It is proved that the field generated by KF in any Mal'cev-Neumann embedding for KF is the universal field of fractions $U(KF)$ of KF. Some consequences are noted. An example is constructed of an embedding $KF \subset D$ into a field D with $D \not\approx U(KF)$. It is also proved that the generalized free product of two free groups can be embedded in a field.

I. Introduction. P. M. Cohn has recently shown [3, Chapter 7] that if R is a semifir then there is an embedding of R in a (not necessarily commutative) field $U(R)$ which is universal in the sense that if $R \subset D$ is another embedding of R in a field then there is a specialization of $U(R)$ onto D which extends the identity map of R. In particular, free associative algebras and free group algebras have universal fields of fractions.

Let now F be a free group and K a commutative field. I. Hughes [5] singles out a class of "free" embeddings (see definition below, §II) of KF into fields and shows that any two free embeddings which are both generated (qua fields) by KF are KF-isomorphic This makes it plausible that $U(KF)$ is a free embedding and we show that this is indeed the case. Oddly enough this is not proved by verifying directly the freedom property of $U(KF)$, but by first proving a subgroup theorem: If G is a subgroup of F, then KG generates, in $U(KF)$, its universal field $U(KG)$.

The significance of our theorem is that it shows that $U(KF)$ is in fact the field generated by KF in any Mal'cev-Neumann embedding of KF [10]. If R is a free K-algebra on the generators of F, then it is easily seen that $U(R) = U(KF)$. Thus we have both $U(R)$ and $U(KF)$ represented in power series over F. This has several interesting consequences: $U(F)$ can be ordered; the center of $U(F)$ is K (if F is not commutative); there is a homomorphism of the multiplicative group $U(F)^*$ onto the free group F. (F is actually a retract of $U(F)^*$.)

Going back to groups, we show that any generalized free product G of two free groups can be embedded in a field. However, using an example of M. Dunwoody [4], we show that there need not exist a fully inverting (definition below) embedding for G, even if the amalgamated subgroup is cyclic.

Hughes [5] asks whether there exists a nonfree embedding of the free group algebra KF. We close by exhibiting such an embedding.

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II. $U(KF)$ **is Hughes-free.** If R is a semifir, we denote by $U(R)$ the universal field of fractions of R. $R \subseteq U(R)$ is fully inverting: every full R-matrix inverts over $U(R)$. Every element u_1 of $U(R)$ is rational over R, i.e. u_1 is the first component of a solution $\mathbf{u}$ of a matrix equation $A\mathbf{u} + \mathbf{a} = 0$ where A is a full $n \times n$ R-matrix and $\mathbf{a} \in R^n$. (Recall that A is full if A is not a product of two matrices of smaller size.) If S is a subring of R, the inclusion $S \to R$ is honest if every full S-matrix is still full as an R-matrix.

If $h: R \to D$ is a homomorphism into a field D, then there is universal specialization $p: U(R) \to D$ which extends h. The domain of p consists of the set of entries of inverses of those R-matrices whose image is invertible over D.

Details and proofs may be found in Cohn [3, Chapter 7].

In particular, if F is a free group and K a commutative field, then KF is a semifir so the above results apply. We write $U(F)$ for $U(KF)$.

If H is a sugroup of F and KF is embedded in a field D, we denote by $\mathrm{Div}_D(H)$ the smallest subfield of D which contains H and K. Note that $\mathrm{Div}_D(H)$ is rational over H.

Our aim is to show that for any subgroup G of a free group F, $\mathrm{Div}_{U(F)}(G) = U(G)$.

The universal specialization from $U(G)$ into $U(F)$ will be a monomorphism $U(G) \to U(F)$ provided the inclusion $KF \to KG$ is honest. It is this that we shall prove. We first deal with a special case.

Lemma 1. *Let F be a free group and G a normal subgroup of finite index in F. Then the inclusion $KG \to KF$ is honest.*

Proof. Let $s_1 = 1, s_2, \ldots, s_n$ be a set of coset representatives for G in F. Then $KF = \bigoplus_{i=1}^{n} (KG)s_i$. Right multiplication by an element of KF is a left KF, and hence a left KG, module homomorphism. Thus we have a faithful representation $\varphi: KF \to (KG)_n$, the $n \times n$ matrices over KG.

If $v \in KG$, then $s_i v = s_i v s_i^{-1} \cdot s_i = v^{s_i} \cdot s_i$. Since G is normal in F, $v^{s_i} \in KG$. Thus $v\varphi$ is the diagonal matrix

$$v\varphi = \begin{bmatrix} v & & & & 0 \\ & v^{s_1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & v^{s_n} \end{bmatrix}. \tag{1}$$

In the obvious way, we extend φ to matrices over KF. Let now M be a KG-matrix which is full over KG. Since conjugation by s_i is an automorphism of KG, M^{s_i} is also full over KG for $i = 1, \ldots, n$. KG is again a free group algebra, so KG is a fir. It follows [3, Theorem 6.4, p. 282] that the diagonal sum $N = M \dotplus M^{s_1} \dotplus \cdots$

$\dotplus M^{s_n}$ is full. But, by (1), $M\varphi$ is similar, via a permutation matrix, to N. So M, as a KF-matrix, maps under φ to a full matrix. So M is full over KF and the lemma is proved.

The next step is to drop the normality assumption on G. We first need an easy lemma.

If S is a subset of a K-algebra, denote by $K\langle S\rangle$ the subalgebra it generates.

Lemma 2. *Let G be a normal subgroup of finite index in the group F, and suppose KF is embedded in a field D. Then* $\mathrm{Div}_D(F)$ *has finite dimension as a left* $\mathrm{Div}_D(G)$ *vector space. Further* $\mathrm{Div}_D(F) = K\langle \mathrm{Div}_D(G), F\rangle$.

Proof. Let $s_1 = 1, s_2, \ldots, s_n$ be a set of coset representatives for G in F, and consider $A = \bigoplus_{i=1}^{n} \mathrm{Div}_D(G)s_i$. Now s_i induces, by conjugation, an automorphism of $\mathrm{Div}_D(F)$ which leaves G, and hence $\mathrm{Div}_D G$, invariant. Thus for $d \in \mathrm{Div}_D(G)$, $d^{s_i} \in \mathrm{Div}_D(G)$. Thus since $s_i d = d^{s_i}s_i$ and $s_i s_j = g_{ij}s_k$ for some k and some $g_{ij} \in G$, A is a K-algebra. Since A has finite left $\mathrm{Div}_D(G)$ dimension, and A is an integral domain, A is a field. Since A contains F, $A = \mathrm{Div}_D(F)$. Clearly A is generated by $\mathrm{Div}_D(G)$ and F. □

Lemma 3. *Let F be a free group, and L a subgroup of finite index in F. Then* $\mathrm{Div}_{U(F)}(L) = U(L)$.

Proof. Let G be the intersection of the conjugates of L. Then G is still of finite index in F and is of course normal. Then, by Lemma 1, $\mathrm{Div}_{U(F)}(G) = U(G)$ and $\mathrm{Div}_{U(L)}(G) = U(G)$.

Let p be the universal specialization p: $U(L) \to \mathrm{Div}_{U(F)}(L)$. Since a full KG-matrix inverts over $U(G)$ it inverts over both $U(L)$ and $\mathrm{Div}_{U(F)}(L)$. So the entries of inverses of full KG-matrices are in the domain $\mathfrak{L}$ of p. So $U(G) \subseteq \mathfrak{L}$. Since p is an L-specialization, also $L \subseteq \mathfrak{L}$. So, by Lemma 2, $U(L) = K\langle U(G), L\rangle \subseteq \mathfrak{L}$. Since p is onto $\mathrm{Div}_{U(F)}L$, p is an L-isomorphism $\mathrm{Div}_{U(F)}L \simeq U(L)$.

The next two lemmas allow us to go from subgroups of finite index to arbitrary finitely generated subgroups.

Lemma 4. *Suppose the free group F is the free product $H_1 * H_2$ of two subgroups. Then* $\mathrm{Div}_{U(F)}(H_1) = U(H_1)$.

Proof (P. M. Cohn). Embed $KH_1 *_K KH_2$ in $U(H_1) *_K U(H_2)$. If M is a KH_1-matrix which is full over KH_1, then M inverts over $U(H_1)$ and hence over $U(H_1) * U(H_2)$. So M is full over $KH_1 * KH_2 = KF$. So M inverts over $U(F)$ and hence inverts over $\mathrm{Div}_{U(F)}(H_1)$. □

Lemma 5. *Let H be a finitely generated subgroup of the free group F. Then H is a free factor in a subgroup L of finite index in F.* □

A proof may be found in [6].

Theorem 1. *If H is a subgroup of the free group F, then* $\mathrm{Div}_{U(F)}(H) = U(H)$.

Proof. If H is finitely generated, this is the contents of Lemmas 3, 4 and 5. Let now H be an arbitrary subgroup of F, and let M be a KH-matrix which is full over KH. Then there is a finitely generated subgroup H' of H such that M is a matrix over KH'. M is still full over KH'. Thus, by the finitely generated case, M inverts over $U(F)$, and hence over $\mathrm{Div}_{U(F)}(H)$. □

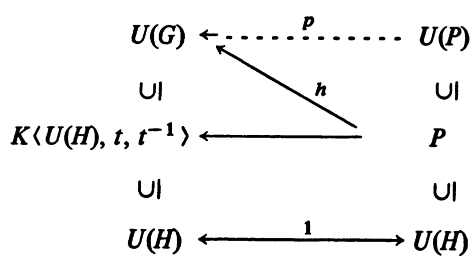
Recall that a group G is indexed at (H,t) if H is normal in G and G/H is the infinite cyclic group generated by the coset tH. Let F be a free group and $KF \subset D$ an embedding of KF in a field. I. Hughes [5] makes the following definition: the embedding $KF \subset D$ is free if for any finitely generated subgroup G of F, and indexing (H,t) of G, then the powers of t are left $\mathrm{Div}_D(KH)$-independent. He then shows

Hughes' theorem. *If $KF \subset D_1$, $KF \subset D_2$ are two free embeddings of KF then there is a KF-isomorphism* $\mathrm{Div}_{D_1}(KF) \approx \mathrm{Div}_{D_2}(KF)$. □

Proposition 6. *$KF \subset U(F)$ is a Hughes-free embedding.*

Proof. Let G be a finitely generated subgroup of F, (H,t) an indexing of G. Since H is normal in G, conjugation by t induces an automorphism τ of $U(H) \subseteq U(F)$. Form the skew Laurent polynomial ring $P = U(H)[z,z^{-1}]$ with the commutation rule $dz = z(d\tau)$. Then P is an Ore domain with quotient field D. Since P is a principal ideal domain, the embedding $P \subset D$ is fully inverting so that $D = U(P)$.

Now we have a homomorphism φ: $P \to K\langle U(H),t,t^{-1}\rangle$ which maps z to t and hence, composing with the inclusion $K\langle U(H),t,t^{-1}\rangle \to U(G)$, a homomorphism h: $P \to U(G)$.



h then extends to a specialization p: $U(P) \to U(G)$. But, identifying $KG = K\langle H,t,t^{-1}\rangle$ with $K\langle H,z,z^{-1}\rangle$, p is a KG-specialization. Since $U(G)$ is fully inverting for KG, so is $U(P)$ (Cohn [3, Theorem 2.3, p. 257]). Hence $U(G)$ and $U(P)$ are KG isomorphic. Clearly, a KG-isomorphism is the identity on H, and hence maps $U(H)$ onto itself. Since $\{z^i\}$ is left $U(H)$ independent, $\{t^i\}$ is left $U(H)$ independent and the proposition is proved.

III. A series representation and applications. We first note that if $R = K\langle X\rangle$ is the free K-algebra on a set X and F is the free group on X, then $U(R) = U(F)$;

since $x \in X$ has an inverse in $U(R)$, there is a homomorphism $KF \to U(R)$ which is the identity on R. This homomorphism extends to a specialization p: $U(F) \to U(R)$. So every matrix over R which inverts over $U(R)$ inverts over $U(F)$, i.e. every full R-matrix inverts over $U(F)$. Since $U(F)$ is generated by R, p is an isomorphism.

Recall the Mal'cev-Neumann method for embedding a free group algebra KF in a field [10]: order F and let D be the set of formal series over F, with coefficients in K, whose support is well ordered. If $0 \neq p \in D$, then p can be written uniquely as $p = kf(1 + p')$, $k \in K$, $f \in F$, $p' = 0$ or p' with positive support. Then $p^{-1} = (1 - p' + p'^2 - \cdots)g^{-1}k^{-1}$. Thus if H is a subgroup of F, $\mathrm{Div}_D(H)$ consists of power series whose support is in H. This makes it clear that D is Hughes-free. Thus, applying Hughes' theorem and the proposition, we obtain

Theorem 2. *Let F be a free group on the set X, $R = K\langle X\rangle$ the free K-algebra on X; D any Mal'cev-Neumann embedding of F. Then* $U(R) = U(F) \simeq \mathrm{Div}_D(F)$. □

Theorem 3. *If R is a free algebra over the ordered (commutative) field K, then $U(R)$ can be ordered.*

Proof. We need only note that a Mal'cev-Neumann field can be ordered if K can be. □

Theorem 4 (cf. Klein [8]). *If R is a noncommutative free K-algebra, then the center of $U(R)$ is K.*

Proof. Let R be freely generated by X, $|X| > 1$, and let F be the free group on X. We may consider $U(R)$ as embedded in a Mal'cev-Neumann field for F. Let $z = \sum_{f \in F} k_f f$ be in the center of $U(R)$, and say f_1 is the least element in the support of z. Then, for $x \in X$, xf_1 and $f_1 x$ are the least elements in the supports of xz and zx. So f_1 is in the center of F, i.e. $f_1 = 1$. So $k_1 - z$ is again in the center of $U(R)$. But then $k_1 - z = 0$ or its support consists of positive elements. This last leads to a contradiction and hence $z \in K$. □

Theorem 5. *Let R be a free K-algebra, F the corresponding free group, and let $U(R)^*$ be the multiplicative group of nonzero elements of $U(R)$. Then the free group F is a retract of $U(R)^*$.*

Proof. We regard $U(R)$ as embedded in a Mal'cev-Neumann field for F. Let N be the subset of $U(R)^*$ of elements $k + P$, where $0 \neq k \in K$, and $P = 0$ or P has positive support. An easy calculation shows that N is normal in $U(R)^*$. If g_1, g_2 are different elements of F, then $g_1 g_2^{-1} \neq 1$. So $g_1 g_2^{-1} \notin N$ and thus $g_1 \neq g_2 \bmod N$. Also, if $Q \in U(R)^*$, Q can be written uniquely as $Q = gk(1 + Q')$ with $k(1 + Q') \in N$, $g \in F$. Then $Q = g \bmod N$ and $Q \to g$ is the required retraction of $U(R)^*$ onto F. □

Corollary. *Let $U(R)^*_{ab}$ be the commutator factor group of $U(R)^*$. The projection $U(R)^* \to U(R)^*_{ab}$ is injective on the generators of R.* □

(This provides a partial answer to problem 10 on p. 286 of [3].)

IV. **Generalized free products of free groups.** Recall from [3] the following fundamental property of free products of rings over a (skew) field D. Let R_1, R_2 be D-rings and let $\{1\} \cup S_i$ be a left D-basis for R_i. Then the monomials on $S_1 \cup S_2$, no two successive letters of which are in the same factor, form, together with 1, a left D-basis for the free product $R_1 *_D R_2$.

Theorem 6 (cf. [1, Corollary 3.1], [7, Theorem 9], [9]). *Let F_1, F_2 be two free groups with a common subgroup H and let G be the generalized free product $F_1 *_H F_2$. Then the group algebra KG can be embedded in a field.*

Proof. If $\{1\} \cup S_i$ is a left transversal for H in F_i then it is clear by looking in Mal'cev-Neumann fields that $\{1\} \cup S_i$ is a left $\mathrm{Div}_{U(F_i)}(H)$-independent set. Further there are KH isomorphisms $\mathrm{Div}_{U(F_1)}(H) \simeq \mathrm{Div}_{U(F_2)}(H) \simeq U(H)$. These observations and the remark preceding the theorem show that the free product $R = U(F_1) *_{U(H)} U(F_2)$ makes sense and embeds KG. But R is a free ideal ring [2] and so has a universal field of fractions $U(R)$. Thus $KG \subseteq U(R)$. □

Unfortunately, $U(R)$ need not be fully inverting for KG. Indeed KG need not have a fully inverting embedding. For Dunwoody [4] has shown that if $G = \langle a, b; a^2 = b^3 \rangle$, then KG has a nonfree finitely generated projective module P. Such a ring has a full matrix which is not invertible in any field which embeds it: let M be a free module of least rank such that $M = P_1 \oplus P$. The projection $M \to P$ is given by an idempotent matrix μ which is not the identity. Thus μ does not become invertible in any overfield. However, μ is full. For otherwise $P \subseteq N$, a submodule of M with fewer generators. Since P is a direct summand of M, it is a direct summand of N, and hence needs fewer generators than M, contradicting the minimality of the rank of M.

V. **An example.** We now construct a nonfree embedding of a free group algebra in a field.

Let F be the free group $F_1 * F_2$ where F_1 is free on z and F_2 is free on x and w. We embed kF_i in $R_i = U(F_i)$. In R_1 we choose a K-basis $\{1\} \cup S_1$ such that $\{(1+z)^{-1}, z^i; i = \pm 1, \pm 2, \ldots\} \subset S_1$ and in R_2 we choose a K-basis $\{1\} \cup S_2$ with $F_2 \backslash \{1\} \subset S_2$. Let $b = (1+z)^{-1}$. In $R = R_1 *_K R_2$ let T' be the set of elements $r = f_1 b f_2 b \cdots f_n b f_{n+1}$ where $f_i \in F$ is a reduced word which neither begins or ends with $z^{\pm 1}$ for $i = 2, \ldots, n-1$, $f_1 = 1$ or f_1 does not end in $z^{\pm 1}$, $f_2 = 1$ or f_2 does not begin with $z^{\pm 1}$. We extend the length function l of F to a length function on $T = T' \cup F\backslash\{1\}$ by declaring $l(r) = n + \sum_{i=1}^{n+1} l(f_i)$. It is clear that a set of elements of T that are spelled differently is left K-independent. We embed R in its universal field $U(R)$. In $U(R)$ we may choose a basis $\{1\} \cup S$ with $T \subset S$. Let $u = bw(1+x)$, and let $Q = \mathrm{Div}_{U(R)}(k[u])$. We claim that in $U(R)$ the set F is left Q-independent.

We first note that Q is the field of right quotients of $k[u]$ so that a set is left Q-independent if and only if it is $k[u]$-independent. So we need only show that F is left $k[u]$-independent. We note next that bw and bwx freely generate a free subalgebra of $U(R)$. Let then f_α be distinct elements of F and suppose that there are polynomials $p_\alpha(u)$, not all zero with $\sum p_\alpha(u)f_\alpha = 0$. Choose α such that $p_\alpha(u)$ has maximal u degree, say n, and f_α has maximal length among the f_β for which $p_\beta(u)$ has degree n. Two cases arise.

1. f_α does not begin with x^{-1}. Then $p_\alpha(u)f_\alpha$ gives rise to a term $t = (bwx)^{n-1}bwxf_\alpha$. This term has length $4n + l(f_\alpha)$ and has degree n on b. Also it is clear that this length is maximal among the monomials in the expansion of $\sum p_\alpha(u)f_\alpha$ which have degree n on b. So since the sum vanishes, for some $\beta \neq \alpha$, $p_\beta(u)$ has degree n and the term t also appears in the expansion of $u^n f_\beta$. Since f_α had maximal length, this forces $f_\alpha = f_\beta$, a contradiction.

2. We may then assume that f_α starts with x^{-1}. Now $p_\alpha(u)f_\alpha$ gives rise to a term $(bwx)^{n-1}bwf_\alpha$ of length $4n + l(f_\alpha) - 1$, and this is the only term of this length in the expansion of $p_\alpha(u)f_\alpha$ (since all other terms in the expansion of $p_\alpha(u)f_\alpha$ either end with x or are too short). It is again easy to see that this implies that $f_\beta = f_\alpha$ for some $\alpha \neq \beta$, and this contradiction proves the claim.

We now provide ourselves with another copy $U(R)'$ of $U(R)$ and consider the free product $V = U(R) * U(R)'$ amalgamating Q with Q'. Then the group words on the letters z, x, w, z', x', w' are left Q-independent. and hence K-independent. Thus the group G generated by these letters is free on them and the K-algebra generated by G in $U(V)$ is the group algebra KG. Now V is still a free ideal ring [2] and hence we may embed V in $U(V)$. Clearly G generates $U(V)$ qua skew fields. However, in $U(V)$,

$$(1+z)^{-1}w(1+x) = (1+z')^{-1}w'(1+x')$$

so that

$$x = w^{-1}(1+z)(1+z')^{-1}w'(1+x') - 1.$$

Hence $\mathrm{Div}(gp(x,z,w,x',z',w')) = \mathrm{Div}(gp(z,w,x',z',w'))$ and $U(V)$ does not distinguish G from a free factor of G. However, it is clear by looking in a Mal'cev-Neumann embedding of KG that if H_1 and H_2 are distinct subgroups of G, then $\mathrm{Div}_{U(G)}(H_1) \neq \mathrm{Div}_{U(G)}(H_2)$. Thus $G \subseteq U(V)$ is not a free embedding.

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